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Simple Toda lattice motions and their linear wave equations

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Abstract. A structure-preserving bijection $B: M \rightarrow W$ from the set M of Toda lattice motions to a natural partition W of the set of linear wave equations in $1+1$ dimensions is applied to obtain some simple and related results about both M and W. The Toda lattice motions derived emphasise the existence of qualitatively distinct Toda lattices depending on the overall sign in the defining dynamical equations. The wave equations that arise illustrate a duality on the set of linear wave equations that is induced by the known duality on M generated by interpolating elements of M into motions of Kac-Van Moerbeke lattices.

1. Introduction

It has been shown (Torrence 1987, hereafter referred to as I) that the family of second-order linear wave equations in $1+1$ dimensions with smooth coefficients has a natural partition, W, whose elements are in a useful correspondence with the set M of motions of the two-dimensional Toda lattice (2DTL). The equivalence relation defining W is based on a generalisation of the Darboux (1882) map between Schrödinger equations, and the bijection $B: M \rightarrow W$, introduced in I, preserves structure by relating that equivalence relation to the dynamical interaction of adjacent elements of the 2DTL. In I $B: M \rightarrow W$ was applied to transfer a known non-trivial result about Toda lattice motions into an apparently new result about linear wave equations. First it was shown that under B each motion of a *finite* 2DTL of N elements *with free ends* goes to an equivalence class of $N-1$ linear wave equations in $1+1$ dimensions, each with the appealing property that *its general solution is a progressing wave of finite order*, as defined by Friedlander (1975). Then from the explicit formulae describing a general solution for the motion of a finite 2DTL with free ends obtained by Leznov and Saveliev (1981), one generated a large family of wave equations with the physically and mathematically simple property that their general solutions are finite-order progressing waves.

It is our purpose in this paper to investigate and illustrate properties of the correspondence $B: M \rightarrow W$ by applying it to obtain some particular simple and related results concerning both Toda lattice motions and linear wave equations. In the next section we will briefly review the construction of the set of equivalence classes of wave equations W, the Toda lattice motions M, suitably complexified, and the bijection $B: M \rightarrow W$. The wave equations with which we will begin are given in § 3. They all have coefficients depending on just one variable, and therefore it will be motions of the one-dimensional Toda lattices (1DTL) that arise, and as they are known to have progressing-wave general solutions of finite order, so it will be motions of finite 1DTL

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that we generate. We will start with a particular countable set of wave equations related to the D'Alembertian in Minkowski space, and by doing elementary coordinate transformations to this set we will generate several additional countable sets of equations. Then in § 4 we will use $B^{-1}: W \to M$ to obtain from the equivalence class, in W, of the lth equation, in each of our countable sets, a particular motion of the free-ended IDTL of $2l+1$ elements. As the equations with which we begin are all self-adjoint, the resulting lattice motions will all be antisymmetrical about a fixed centre element, an instance of a general property of $B^{-1}: W \to M$ discussed in I.

In § 5 we will pass from each one of these simple IDTL motions to a second motion by utilising the pairing of Toda lattice motions through the Kac–Van Moerbeke lattices (1975) as discussed by Toda (1981). Each motion of one of our lattices of $2l+1$ elements can be paired in this way to a motion of a lattice of $2l$ elements. The new motions are again antisymmetrical, but they comprise an even number of elements and there are no fixed centre elements. In the same section $B: M \to W$ is used to generate from these new IDTL motions corresponding new equivalence classes of wave equations. The duality induced on W by that on M clearly preserves the finite-order progressing-wave property, as it carries *finite* IDTL motions to *finite* IDTL motions. At the same time it does not respect self-adjointness of wave equations. The result is a set of countable families of wave equations that are *not* self-adjoint and have progressing-wave general solutions, a result which may be new. They are dual to the useful, simple wave equations with which we began, and are simple in their own right. Their usefulness, in their own right, remains to be determined. In the concluding § 6 we isolate what we feel may be the more original, or useful, items that have been covered in the body of the paper.

2. The map $B: M \to W$

It is straightforward to verify that by utilising a factor transformation on the dependent variable and coordinate transformations on the two independent variables any homogeneous second-order linear wave equation in $1+1$ dimensions with sufficiently differentiable coefficients can be put into either of the normal forms

$$(\partial_v j_0 \partial_u - j_1)\psi_0 = 0 \tag{2.1a}$$

$$(\partial_u \tilde{j}_0 \partial_v - \tilde{j}_{-1})\tilde{\psi}_0 = 0 \tag{2.1b}$$

where $j_0, j_1 \tilde{j}_0, \tilde{j}_{-1}$ are functions of u and v. Essentially following Kundt and Newman (1968), one can show that inductively defining $j_{n+1}(u, v)$, $\psi_n(u, v)$, and $\tilde{j}_{n-1}(u, v)$, $\tilde{\psi}_n(u, v)$, where n is any integer, by

$$\frac{j_{n+1}}{j_n} = \frac{j_n}{j_{n-1}} - \partial_{uv} \ln|j_n| \qquad \partial_v(j_n\psi_n) = j_n\psi_{n-1} \tag{2.2a}$$

and

$$\frac{\tilde{j}_{n-1}}{\tilde{j}_n} = \frac{\tilde{j}_n}{\tilde{j}_{n+1}} - \partial_{uv} \ln|\tilde{j}_n| \qquad \partial_u(\tilde{j}_n\tilde{\psi}_n) = \tilde{j}_n\tilde{\psi}_{n+1} \tag{2.2b}$$

respectively, leads each of (2.1) into a sequence of equivalent wave equations

$$(\partial_v j_n \partial_u - j_{n+1})\psi_n = 0 \tag{2.3a}$$

$$(\partial_u \tilde{j}_n \partial_v - \tilde{j}_{n-1})\tilde{\psi}_n = 0 \tag{2.3b}$$

respectively. The various sequences of j, ψ and wave equations will be referred to indiscriminately as *substitution sequences.* Naturally the different normal forms are related, and the simple connection is

$$\tilde{j}_n j_n = 1 \qquad \tilde{\psi}_n = j_n \psi_n. \tag{2.4}$$

Regardless of which normal form one considers, the corresponding substitution sequence is generated in both directions, and it clearly terminates for both sequences in both directions if and only if there exist integers M and N such that the equivalent conditions

$$0 = j_{N+1} = 1/j_{M-1} \qquad 0 = \tilde{j}_{M-1} = 1/\tilde{j}_{N+1} \tag{2.5}$$

are satisfied. But precisely in this case the *terminating* substitution sequences of j correspond to *terminating* substitution sequences of wave equations which end with

$$\partial_v j_N \partial_u \psi_N = 0 \qquad \partial_u \tilde{j}_M \partial_v \tilde{\psi}_M = 0. \tag{2.6}$$

It is easy to solve each of (2.6), and then from (2.2) to solve (2.1), obtaining

$$\psi_0 = \tilde{j}_1 \partial_v (j_1 \tilde{j}_2) \ldots \partial_v (j_{N-1} j_N) \partial_v (j_N a) \tag{2.7a}$$

and

$$\tilde{\psi}_0 = j_{-1} \partial_u (\tilde{j}_{-1} j_{-2}) \ldots \partial_u (\tilde{j}_{M+1} j_M) \partial_u (\tilde{j}_M b) \tag{2.7b}$$

respectively, with $a = a(v)$, $b = b(u)$ arbitrary sufficiently differentiable functions. Thus one finds from (2.4) that

$$\psi = \psi_0 + \tilde{j}_0 \tilde{\psi}_0 \qquad \text{and} \qquad \tilde{\psi} = \tilde{\psi}_0 + j_0 \psi_0 \tag{2.8}$$

solve (2.1a) and (2.1b) respectively. But then (2.8) provides the general solutions of (2.1), and these are obviously progressive waves of finite order (and expressible, consequently, in closed form).

The infinite Toda (1981) lattice, generalised to two dimensions, is the dynamical system governed by the countable set of coupled non-linear partial differential equations

$$\partial^2_{uv} \boldsymbol{r} \equiv (\partial_t^2 - \partial_x^2) \boldsymbol{r} = \varepsilon K_\infty \mathrm{e}^{-\boldsymbol{r}} \tag{2.9}$$

where $t \equiv v + u$, $x \equiv v - u$, $(\mathrm{e}^{-\boldsymbol{r}})_i \equiv \mathrm{e}^{-r_i}$, $-\infty < i < +\infty$, $\varepsilon = \pm 1$, and

$$K_\infty \equiv \begin{bmatrix} \ddots & & & & & & & \\ 0 & -1 & 2 & -1 & 0 & & & \\ & 0 & -1 & 2 & -1 & 0 & & \\ & & & -1 & 2 & -1 & 0 & \\ & & & & & & \ddots & \end{bmatrix}. \tag{2.10}$$

The choice of ε is related to Toda (1981) by $\varepsilon = |ab|/ab$, where a, b are the two parameters in the dynamical equations in his § 2.2. Changing the signs of both a and b is equivalent to merely taking $\boldsymbol{r} \to \boldsymbol{r}' = -\boldsymbol{r}$. However the sign of ε is physically significant. Toda consistently assumed that $\varepsilon = 1$. However in § 4 we will encounter both choices and for $\varepsilon = -1$ IDTL motions qualitatively different from those usually considered.

By a *finite* 2DTL of N elements with free ends is meant the double truncation of (2.9) and (2.10) given by taking $\boldsymbol{r}=\{r_i\}_{i=1,\ldots,N-1}$ and replacing K_∞ by the $(N-1)\times(N-1)$ matrix

$$K_{N-1}=\begin{bmatrix} 2 & -1 & 0 & & & & \\ & \ddots & & & & & \\ & 0 & -1 & 2 & -1 & 0 & \\ & & & & \ddots & & \\ & & & & 0 & -1 & 2 \end{bmatrix}. \tag{2.11}$$

The truncated system with $N-1$ degrees of freedom is said to govern a lattice with N elements because in the usual mechanical interpretation of these dynamical systems r_i is understood to be the *relative* displacement, $r_i=y_{i+1}-y_i$, of 'adjacent' elements of the lattice.

Actually, a mechanical interpretation of (2.9) for the finite-dimensional case deserves some discussion. Assume we have a possibly infinite set of parallel strings arrayed parallel to the x axis, orthogonal to the z axis, and free to vibrate in the y direction with a displacement $y_i(x,t)$. If the relative displacement of adjacently indexed strings is given by $r_i=y_{i+1}-y_i$, and each point of the ith string interacts exponentially with the corresponding (i.e. same x and t) points of the $(i+1)$th and $(i-1)$th strings, then (2.9) governs the motion of the system. If $\varepsilon=-1$ in (2.9), the *magnitude* of the force on the ith string due to the $(i+1)$th string drops exponentially from $+\infty$ to 0 as $y_{i+1}-y_i$ rises from $-\infty$ to $+\infty$, and similarly for the effect of the $(i-1)$th string. The *direction* of the force due to the $(i+1)$th ($(i-1)$th) string is always in the negative (positive) y_i direction. For $\varepsilon=+1$ an obvious 'reversal' of the directional remark applies. For either sign of ε the insensitivity of the *direction* of the force between adjacently indexed strings to the *sign* of their relative displacement is interpretatively awkward. For the infinite lattice this awkwardness can be avoided by assuming, as did Toda (1981), an additional constant force between adjacently indexed strings that opposes the exponential force. These extra forces disappear in pairs from the dynamics of the infinite lattice, but can be viewed as creating an equilibrium relative displacement for each adjacent pair, and this explains why (2.9) and (2.10) can be satisfied by a time-independent set of relative displacements. Unfortunately with the finite lattice (2.9) and (2.11), our concern in this paper, this interpretive device is not available. The end strings, i.e. those with minimum and maximum index values, are subject to non-zero forces whose directions never change, and they can have no equilibrium position. When we restrict ourselves to the finite 1DTL by assuming $\partial r_i/\partial x=0$ for all i, these interpretive peculiarities remain, and account for the well known results of Moser (1975) for the case $\varepsilon=+1$ that all finite 1DTL motions produce infinite displacements and constant velocities as $t\to\pm\infty$. On the other hand, for $\varepsilon=-1$ we will encounter examples of three distinct types of motion, all of them different from the qualitatively homogeneous $\varepsilon=+1$ motions.

We now define W by the equivalence relation embodied in (2.3a). We define M to be all solutions of (2.9) with either (2.10) or (2.11), with the understanding that any of the displacements y_i may have an additive imaginary part $\sqrt{-1}\pi$. Then

$$B: M\to W:\{y_i\}\to\{j_i\} \qquad j_i=\mathrm{e}^{-y_i} \tag{2.12}$$

is well defined and a bijection. The essential results of I are that B, so defined, carries sets of $\{y_i\}$ that generate $r_i = y_{i+1} - y_i$ that satisfy (2.9) with $\varepsilon = -1$, to sets of $\{j_i\}$ related by the defining equations, (2.2), of the substitution sequences, and that free-ended finite 2DTL are mapped to doubly terminating substitution sequences. It is a fact that the sequences can have elements of both signs and it is this that obliges us to complexify M as we did above. We will find that the specific sequences that we encounter in § 4 comprise positive j, which pose no problem, or sequences that alternate in sign, which is easily seen to be equivalent to a real motion with $\varepsilon = +1$.

3. Some simple wave equations

If one begins with the usual covariant scalar wave equation $\psi = 0$ on Minkowski space, expresses it in spherical coordinates T, r, θ, φ, and assumes that $\psi = \psi_l(T, r)Y_l^m(\theta, \varphi)/r$ where the Y_l^m are the usual spherical harmonics, one finds that the $\psi_l(t, x)$ must satisfy

$$\partial_{uv}^2\psi_l \equiv (\partial_t^2 - \partial_x^2)\psi_l = \frac{l(l+1)}{t^2}\psi_l \equiv \frac{l(l+1)}{(v+u)^2}\psi_l \tag{3.1}$$

where $t = r = v+u$, $x = T = v-u$, and l is any non-negative integer. That equations (3.1) have progressing-wave solutions is known (see for example Courant and Hilbert (1961)), though rarely emphasised. It is clear that the transformation $u = u(U)$, $v = v(V)$ leaves (3.1) in normal form and that the resulting equations

$$\partial_{UV}^2\psi_l = \frac{(\partial v/\partial V)(\partial u/\partial U)l(l+1)}{[u(U)+v(V)]^2}\psi_l \tag{3.2}$$

also have finite-order progressing-wave solutions since (3.1) does. Despite the elementary connection between the sets of wave equations, (3.1) and (3.2), we will find that the corresponding Toda lattice motions can be qualitatively different. Particular choices of $v(V)$ and $u(U)$ yield all the wave equations that we will work with in the next section. If we choose $u = \tanh(aU)$, $v = \tanh(aV)$, or $u = \tan(aU)$, $v = \tan(aV)$, where a is any positive constant, then in (3.2) we obtain

$$(\partial_t^2 - \partial_x^2)\psi_l = \frac{a^2 l(l+1)}{\sinh^2(at)}\psi_l \tag{3.3}$$

$$(\partial_t^2 - \partial_x^2)\psi_l = \frac{a^2 l(l+1)}{\sin^2(at)}\psi_l \tag{3.4}$$

respectively. A third choice, $u = \tanh(aU)$, $v = \coth(aV)$, results in

$$(\partial_t^2 - \partial_x^2)\psi_l = -\frac{a^2 l(l+1)}{\cosh^2(at)}\psi_l. \tag{3.5}$$

In the latter set the right-hand sides are the well known Bargmann reflectionless potentials. It is not surprising that in a wave equation context they are associated with finite-order progressing waves; however, their simple connection, through transformations in the uv plane, with the familiar potentials $l(l+1)/t^2$ is rarely mentioned.

For (3.1) and (3.3)-(3.5) it is easy to write down the substitution sequences. We note that in all four cases the equations are already in normal form, and with $j_0 = 1$, which implies that $j_i = 1/j_{-i} = \tilde{j}_{-i}$. The four sequences begin with $j_0 = 1$ and

$$j_1 = \begin{cases} l(l+1)/t^2 \equiv l(l+1)/R_1^2(t) & (3.6) \\ a^2 l(l+1)/\sinh^2(at) \equiv l(l+1)/R_2^2(t) & (3.7) \\ a^2 l(l+1)/\sin^2(at) \equiv l(l+1)/R_3^2(t) & (3.8) \\ -a^2 l(l+1)/\cosh^2(at) \equiv l(l+1)/R_4^2(t) & (3.9) \end{cases}$$

respectively. In all four cases it is easy to check that $\partial_{uv} \ln|j_1| = 2j_1/l(l+1)$. That, with $j_0 = 1$, allows one to easily confirm that the full substituting sequences are simply given by

$$\begin{aligned} j_k &= R_\delta^{-2k} \prod_{i=1}^{k} [l(l+1) - i(i-1)] = \frac{1}{j_{-k}} \qquad 1 \leqslant k \leqslant l \\ j_{l+1} &= \frac{1}{j_{-l-1}} = 0 \qquad j_0 = 1 \end{aligned} \tag{3.10}$$

with $\delta = 1, 2, 3, 4$. It should be noted that for $\delta = 1, 2, 3$ the j_k are all positive, but for $\delta = 4$ they alternate in sign. We see that the ratios of consecutive j, which figure prominently in the solutions of (3.1) and (3.3)-(3.5), according to (2.7), are simply

$$\frac{j_{k+1}}{j_k} = R_\delta^{-2} \frac{l(l+1) - (k+1)k}{l(l+1) - k(k-1)}. \tag{3.11}$$

4. Some simple Toda lattice motions

We now apply B^{-1}: $W \to M$ to the $\{j_k\}$ given by (3.10), first for each of (3.6)-(3.8). The resulting displacements and velocities are $y_0 = \dot{y}_0 = 0$, and for $1 \leqslant k \leqslant l$,

$$y_k = 2k \ln|t| - K(l, k) \qquad \dot{y}_k = 2k/t \tag{4.1}$$

$$y_k = 2k \ln\left|\frac{\sinh(at)}{a}\right| - K(l, k) \qquad \dot{y}_k = 2ka \coth(at) \tag{4.2}$$

$$y_k = 2k \ln\left|\frac{\sin(at)}{a}\right| - K(l, k) \qquad \dot{y}_k = 2ka \cot(at) \tag{4.3}$$

with $y_{-k} = -y_k$ and

$$K(l, k) \equiv \sum_{i=1}^{k} \ln[l(l+1) - i(i-1)] \qquad 1 \leqslant k \leqslant l. \tag{4.4}$$

These are real solutions of (2.9) and (2.11) for $\varepsilon = -1$. The time intervals on which (4.1)-(4.3) apply are $(-\infty, 0)$ or $(0, \infty)$ for (4.1) and (4.2), and $(-\pi/a, 0)$ or $(0, \pi/a)$ for (4.3). The motions on the negative intervals are obviously the time reversals of those on the positive intervals in all three cases, and the motions are all antisymmetrical in the sense that $y_{-k} = -y_k$, $y_0 = 0$. Thus in discussing the motions we will just refer to the y_k, $0 < k \leqslant l$, with $t > 0$.

Beginning with (4.1) we see that the l particles 'start' from $y_k = -\infty$ with $\dot{y}_k = +\infty$ when $t = 0$, reach $y_k = -K(l, k)$ with $\dot{y}_k = 2k$ when $t = 1$, and approach $y_k = +\infty$ with $\dot{y}_k$ approaching 0 when $t \to +\infty$. In the case of (4.2) the particles begin from the same place, reach the displacements $-K(l, k)$ sooner, and approach $y_k = +\infty$ as $t \to \infty$, but with asymptotic velocities of $2ka > 0$. Finally for (4.3) the particles start similarly, but slow to a halt when $t = \pi/2a$, and then return to $y_k = -\infty$ with $\dot{y}_k$ approaching $-\infty$, as $t \to \pi/a$. The three quite different motions can be intuitively understood if one thinks of the whole system as a single particle moving next to a wall of potential of the form $-e^{-r}$, which thus ranges from $-\infty$ at $r = -\infty$ to an asymptotic maximum of 0 as r goes to $+\infty$. In the case of (4.3) we are looking at a *bound state* where the particles have insufficient energy to escape to $y = +\infty$, and must fall back to $y = -\infty$. For (4.1) we have, in a sense, an *equilibrium state* where the particles have exactly the energy required to escape to $y = +\infty$ with no residual kinetic energy when they get there. Lastly with (4.2) we have a *free state* as the particles can reach $y = +\infty$ with energy (velocity) to spare. These remarks are consistent with a calculation of the system energies. For (4.2) and (4.1), summing the kinetic energies at $y_k = +\infty$ where the potential energies vanish gives, respectively,

$$E_2 = 2 \sum_{k=1}^{l} \tfrac{1}{2}(2ka)^2 = \tfrac{2}{3}l(l+1)(2l+1)a^2 \qquad E_1 = 0. \tag{4.5}$$

For (4.3) the direct calculation is not quite as simple since we would have to consider the potential energies of relative displacement at $t = \pi/2a$ when the particles are all momentarily at rest; however, noting that substituting $a\sqrt{-1}$ for a in (4.2) produces (4.3) it follows that the answer must be

$$E_3 = -\tfrac{2}{3}l(l+1)(2l+1)a^2 \tag{4.6}$$

in this case. These three motions are not among those discussed by Toda (1981) for the finite IDTL following Moser (1975) because these authors were solving (2.9) and (2.11) with $\varepsilon = +1$ while these are solutions for $\varepsilon = -1$.

Applying $B^{-1}: W \to M$ to (3.10) as generated from (3.9) results in the j alternating in sign, and consequently we must add $\pi\sqrt{-1}$ to each odd indexed y_k, where the *real parts* of *all* the y_k are $y_0 = 0$,

$$\operatorname{Re}(y_k) = 2k \ln\left|\frac{\cosh(at)}{a}\right| - K(l, k) \qquad \dot{y}_k = 2ka \tanh(at) \tag{4.7}$$

$k = 1, \ldots, l$, with $y_{-k} = -y_k$. Substituting these alternately real and complex y_k into (2.9) and (2.11) shows that the real parts, precisely (4.7), are a solution for $\varepsilon = +1$. The formulae apply for the time interval $(-\infty, +\infty)$, with the motion again antisymmetrical. This time the $1 \leq k \leq l$ particles start from $y_k = +\infty$ with velocities of $-2ka$ when $t = -\infty$, move with decreasing speed in the direction of decreasing y_k, come to momentary rest at $t = 0$, and then return to $y_k = +\infty$ with asymptotic velocities of $+2ka$. This motion can be understood if one views the system as a particle moving beside a potential wall of the form e^{-r}, which is infinitely high at $-\infty$ and approaches the minimum value 0 at $+\infty$. In this case energy must be positive, and one has a bound motion regardless of its exact value. The asymptotic kinetic energies immediately give

$$E_4 = \tfrac{2}{3}l(l+1)(2l+1)a^2 \tag{4.8}$$

for the system energy. As we have an $\varepsilon = +1$ solution for (4.7), this is, as was to be expected, a simple example of the finite Toda lattice motions discussed by Moser (1975).

5. The dual motions and wave equations

By doing a calculation parallel to that in § 3.8 of Toda (1981) it is easy to verify that if we have a solution $\{x_k\}$ of the Kac-van Moerbeke (1975) system

$$\dot{x}_{2k} = \varepsilon \exp[-(x_{2k} - x_{2k-1})] + \exp[-(x_{2k+1} - x_{2k})] \tag{5.1}$$

$$\dot{x}_{2k+1} = \exp[-(x_{2k+1} - x_{2k})] + \varepsilon \exp[-(x_{2k+2} - x_{2k+1})] \tag{5.2}$$

then the sets $\{x_{2k}\}$ and $\{x_{2k+1}\}$, appropriately reindexed, are themselves, separately, solutions of (2.9). This calculation simply generalises Toda's formulae to include the case $\varepsilon = -1$. As we have three qualitatively distinct sets of solutions of (2.9) and (2.11) for $\varepsilon = -1$, and one set for $\varepsilon = +1$, we can, in principle, generate three additional sets for the $\varepsilon = -1$ case and another set for the $\varepsilon = +1$ case. We will utilise the simplicity of the solutions (4.1)-(4.3) and (4.7) to calculate new, dual, motions explicitly. We begin with the y_k given by (4.1)-(4.3) or (4.7), with the correct value for ε in each case. In terms of the Kac-Van Moerbeke lattice we known $x_{2k} = y_k$, and by substitution into (5.1) and (5.2) we obtain coupled equations overdetermining the x_{2k+1}. That these equations are consistent with x_{2k+1} that in fact satisfy (2.9) is the interesting duality noted by Kac and Van Moerbeke. Starting from the centre equation with $x_0 = y_0 = 0$, which yields $x_{-1} = -x_1$, we can find a simple set of x_{2k-1}, $-l+1 \leq k \leq l$, that satisfy (5.1) and (5.2). The resulting motion of the Kac-Van Moerbeke lattice is antisymmetrical, with $4l+1$ elements. Thus the new IDTL motion has $2l$ elements and is also antisymmetrical, but there is no (fixed) centre element. All of these remarks apply equally to all four cases, and the four new Toda lattice motions are, for $\varepsilon = -1$,

$$y_m = (2m-1)\ln|t| - M(l, m) \qquad \dot{y}_m = (2m-1)/t \tag{5.3}$$

$$y_m = (2m-1)\ln\left|\frac{\sinh(at)}{a}\right| - M(l, m) \qquad \dot{y}_m = (2m-1)a\coth(at) \tag{5.4}$$

$$y_m = (2m-1)\ln\left|\frac{\sin(at)}{a}\right| - M(l, m) \qquad \dot{y}_m = (2m-1)a\cot(at) \tag{5.5}$$

$$y_m = (2m-1)\ln\left|\frac{\cosh(at)}{a}\right| - M(l, m) \qquad \dot{y}_m = (2m-1)\tanh(at) \tag{5.6}$$

for $\varepsilon = +1$, where $y_{1-m} = -y_m$, $1 \leq m \leq l$, and

$$M(l, m) \equiv \sum_{i=1}^{m} \ln[l^2 - (i-1)^2] - \ln l. \tag{5.7}$$

The interpretive remarks concerning the motions (4.1)-(4.3) and (4.7) carry over verbatim to (5.3)-(5.6), with only the numbers changed.

It remains to apply $B: M \to W$ to (5.3)-(5.6), and we will have generated linear wave equations dual to those, (3.1) and (3.3)-(3.5), with which we began. The first step is to generate the dual substitution sequences, and we obtain from (2.12)

$$\begin{aligned} j_m &= R_\delta^{-(2m-1)} l \prod_{i=1}^{m} [l^2 - (i-1)^2] = 1/j_{1-m} \qquad 1 \leq m \leq l \\ j_{m+1} &= 0 = 1/j_{-m} \end{aligned} \tag{5.8}$$

where the R_δ, $\delta = 1, 2, 3, 4$, are as defined by (3.6)-(3.9). The linear wave equations themselves are for $\varepsilon = -1$

$$\left((\partial_t + \partial_x)\frac{t}{l}(\partial_t - \partial_x) - \frac{l}{t}\right)\psi_l = 0 \tag{5.9}$$

$$\left((\partial_t + \partial_x)\frac{\sinh(at)}{la}(\partial_t - \partial_x) - \frac{la}{\sinh(at)}\right)\psi_l = 0 \tag{5.10}$$

$$\left((\partial_t + \partial_x)\frac{\sin(at)}{la}(\partial_t - \partial_x) - \frac{la}{\sin(at)}\right)\psi_l = 0 \tag{5.11}$$

$$\left((\partial_t + \partial_x)\frac{\cosh(at)}{al}(\partial_t - \partial_x) + \frac{la}{\cosh(at)}\right)\psi_l = 0 \tag{5.12}$$

for $\varepsilon = +1$. The closed form finite-order progressive-wave solutions for these equations, whose existence is the equations' noteworthy property, are of course given by (2.7), where the j were just provided by (5.8). The *ratios* of consecutive j, which occur in (2.7), are simply given by

$$\frac{j_{m+1}}{j_m} = R_\delta^{-2}\frac{l^2 - (m+1)m}{l^2 - m(m-1)}. \tag{5.13}$$

The non-self-adjointness of (5.9)-(5.12) is manifest.

6. Conclusion

The specific calculations contained in §§ 3-5 have referred to very particular motions of IDTL, and very special linear wave equations. These results in themselves seem to us of some interest. The existence of such countable sets of finite IDTL motions indexed by the lengths of the lattices, and of such a simple analytical form, is not available in the literature as far as we know. The search for linear wave equations with finite-order progressing-wave solutions was described by Courant and Hilbert (1961) as a problem 'hardly touched', and the generation of simple sets of such equations, not self-adjoint, may be a worthwhile contribution.

Perhaps of more interest are the insights into general properties of the structures involved that the specific examples provide. First, the qualitative distinction between the $\varepsilon = +1$ and $\varepsilon = -1$ IDTL motions, and in fact such distinctions within those of the $\varepsilon = -1$ type, seems not to have been explicitly discussed before. Those distinctions are sharply made here as we have given simple examples of each type. Second, the convenience of complexifying the family M of lattice motions in order that $B: M \to W$ be a bijection is also made clear as this arises for the $\varepsilon = +1$ family of motions. Although there may be no interest in this from the point of view of Toda lattice motions, it allows one to manipulate equivalence classes of real linear wave equations through their preimages under B, and this may be important. This last consideration is emphasised by what seems to us the most interesting general result illustrated by our specific calculations. The known duality of the set M of lattice motions that obtains when such motions are viewed as interpolated motions within a Kac-Van Moerbeke lattice motion has been shown, by example, to induce a duality on at least some subset of the set W of equivalence classes of linear wave equations. This duality may not have been recognised before and, given the ubiquity of such partial differential equations in mathematical physics, may be useful. The fixed centre antisymmetric elements of M can be related to classical fields of integer spin, as indicated by their generation from a scalar (spin 0) field equation, while it can be shown that there is an analogous tenuous connection between the antisymmetric elements of M without a fixed centre, and field equations for half-integer spins. One speculates that the Kac-Van Moerbeke duality on M, and the induced duality on W, might have antecedents in supersymmetry.

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References

Courant R and Hilbert D 1961 *Methods of Mathematical Physics* vol 2 (Berlin: Springer)
Darboux G 1882 *C.R. Acad. Sci., Paris* **94** 1456
Friedlander F G 1975 *The Wave Equation on a Curved Space Time* (Cambridge: Cambridge University Press)
Kac M and Van Moerbeke P 1975 *Adv. Math.* **16** 160
Kundt W and Newman E T 1968 *J. Math. Phys.* **9** 2193
Leznov A N and Saveliev M V 1981 *Physica* **3D** 62
Moser J 1975 *Adv. Math.* **16** 197
Toda M 1981 *Theory of Nonlinear Lattices* (Berlin: Springer)
Torrence R J 1987 *J. Phys. A: Math. Gen.* **20** 91